Conformal invariance in elementary particle physics

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Fiz. Elem. Chastits At. Yadra 9, 1039–1069 (September-October, 1978)

Some aspects of conformal invariance in elementary particle physics are considered: the laws of conformal transformations of fields, the conformally covariant two-point and three-point functions, the dimensional properties of the energy-momentum tensor, the mass radius of the pion, the $\sigma\pi\pi$ vertex, and the connection between scale and chiral invariance.

PACS numbers: 11.30.Cp, 11.10. – z

INTRODUCTION

In this review, we consider some aspects of conformal invariance in elementary particle physics. The theory of conformal invariance began to attract particular attention after the discovery of scaling[1] in processes of deep inelastic interaction between leptons and hadrons, although it had already been used much earlier in some theoretical calculations.[2-8] As early as 1963, Markov[9] had pointed out for the first time that the total cross sections for inelastic interaction of neutrinos with nucleons could have pointlike behavior. In the meanwhile this idea has been brilliantly confirmed by various experiments.

The important part played by scale transformations in deep inelastic processes was pointed out by Bogolyubov, who emphasized that the form factors of these processes have a behavior very similar to the behavior of similarity solutions in the problem of a strong "point" explosion in classical gas dynamics and hydrodynamics. On the basis of this analogy, Matveev, Muradyan, and Tavkhelidze[10] formulated a similarity (self-similarity) principle, according to which scale invariance is a universal model-independent property of all deep inelastic processes that is governed by the laws of physical similitude and dimensional analysis.

Baldin[11] considered the applicability of scale invariance to the description of relativistic nuclei and predicted the effect of cumulative particle production.[12]

It has been shown[13-15] that in renormalizable theories with no derivatives in the interaction scale invariance (in conjunction with relativistic invariance) entails invariance under a larger group of transformations, the conformal group. From the algebraic point of view, the conformal group is interesting in that it is an extension of the Poincaré algebra to an orthogonal algebra of higher dimension. As a symmetry group of space-time, the conformal group is the most general group that leaves the light cone invariant. Moreover, as Ogievetsky[16,17] has shown, the action of the general covariance group can be reduced to repeated applications of two of its subgroups, namely, the special linear group $SL(4,R)$ and the conformal group. He has also shown that the gravitational field is associated with joint nonlinear realizations of dynamical conformal symmetry and dynamical affine symmetry.

Many interesting ideas in quantum field theory, and also many new directions in the physics of strong interactions, have been formulated and studied by means of the conformal group. The requirement of conformal invariance has made it possible to determine almost uniquely the two- and three-point correlation functions.[18-21] To find many-point ($n \geqslant 4$) Green's functions, Mack and Todorov[22,23] have formulated a "skeleton" diagram technique that is free of ultraviolet divergences.

It is obvious that scale invariance cannot be satisfied exactly because of the discrete nature of the particle mass spectrum. The use of scale invariance in high energy physics is justified by the fact that when the energy of all the particles is much greater than their masses the latter can be taken to be zero with sufficient accuracy. That particles have masses indicates a strong breaking of scale invariance in reality. This means that the currents corresponding to a scale transformation or special conformal transformation cannot be conserved. The nature of the breaking of the scale and conformal invariances is manifested in the expressions for the divergences of these currents.

Finally, we may point out that the assumption of scale (conformal) symmetry is a fruitful physical idea and that the search for correct ways for breaking this symmetry could play a heuristic role in the construction of better theories.

1. THE CONFORMAL GROUP

The conformal group is a 15-parameter continuous group of space-time transformations that include:

(a) the inhomogeneous Lorentz transformations

$$x'_\mu = \Lambda^\nu_\mu x_\nu + a_\mu \; (g^{\mu\nu}\Lambda^\rho_\mu\Lambda^\sigma_\nu = g^{\rho\sigma}); \tag{1}$$

(b) the scale transformation

$$x'_\mu = \rho x_\mu \quad (\rho > 0); \tag{2}$$

(c) the special conformal transformation

$$x'_\mu = (x_\mu + c_\mu x^2)/(1 + 2cx + c^2x^2). \tag{3}$$

We denote the generators of these transformations by $M_{\mu\nu}$, P_μ, D, and K_μ, respectively. They satisfy the commutation relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\,(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}); \tag{4}$$
$$[M_{\mu\nu}, P_\rho] = i\,(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu); \tag{5}$$
$$[P_\mu, P_\nu] = 0; \tag{6}$$
$$[M_{\mu\nu}, D] = 0; \tag{7}$$
$$[M_{\mu\nu}, K_\rho] = i\,(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu); \tag{8}$$
$$[P_\mu, D] = -iP_\mu; \tag{9}$$
$$[P_\mu, K_\nu] = -2i\,(g_{\mu\nu}D + M_{\mu\nu}); \tag{10}$$

$$[D, K_\mu] = -iK_\mu; \quad (11)$$
$$[K_\mu, K_\nu] = 0. \quad (12)$$

Introducing the antisymmetric operators J_{AB} ($A, B = 0, 1, 2, 3, 5, 6$):

$$J_{\mu\nu} = M_{\mu\nu};\ J_{\mu\sigma} = (P_\mu - K_\mu)/2; \\ J_{\mu\sigma} = (P_\mu + K_\mu)/2;\ J_{56} = -D, \quad (13)$$

we can combine the commutation rules (4)–(12):

$$[J_{AB}, J_{CD}] = i\,(g_{AD}J_{BC} + g_{BC}J_{AD} - g_{AC}J_{BD} - g_{BD}J_{AC}), \quad (14)$$

where

$$g_{AB} = \mathrm{diag}(+---;-+).$$

Thus, the conformal algebra is isomorphic to the orthogonal algebra $O(4, 2)$. The conformal group can therefore be regarded as a group of pseudorotations in a six-dimensional space.

We now consider transformations of fields under conformal transformations. From (1)–(3) we see that the point $x = 0$ remains unchanged under homogeneous Lorentz transformations, the scale transformation, and the special conformal transformation—these transformations form the little group of the conformal group. Given any representation of this little group, one can determine the complete action of the generators of the conformal group on the field $\varphi(x)$. This is achieved by the method of the theory of induced representations as follows.[13,24]

Suppose that

$$[M_{\mu\nu}, \varphi_\alpha(0)] = -(\Sigma^{(\varphi)}_{\mu\nu}\varphi(0))_\alpha; \quad (15)$$
$$[D, \varphi_\alpha(0)] = -(\Delta^{(\varphi)}\varphi(0))_\alpha; \quad (16)$$
$$[K_\mu, \varphi_\alpha(0)] = -(k^{(\varphi)}_\mu\varphi(0))_\alpha, \quad (17)$$

where $\Sigma_{\mu\nu}$, Δ, and k_μ are matrices satisfying the same commutation relations as $M_{\mu\nu}$, D, and K_μ. We are interested in finding the rules for the commutators $[J, \varphi_\alpha(x)]$ of elements J of the conformal algebra and the field operator $\varphi_\alpha(x)$. We choose a basis in the space of indices α such that the translation operator P_μ does not affect the indices, i.e.,

$$\exp(iaP)\,\varphi_\alpha(x)\exp(-iaP) = \varphi_\alpha(x+a). \quad (18)$$

Using (18), we can write

$$[J, \varphi_\alpha(x)] = \exp(iPx)\,[J', \varphi_\alpha(0)]\exp(-iPx), \quad (19)$$

where

$$J' \equiv \exp(-iPx)\,J\exp(iPx). \quad (20)$$

Using the commutation relations (5), (6), (9), and (10), we find

$$M'_{\mu\nu} = M_{\mu\nu} + x_\mu P_\nu - x_\nu P_\mu; \quad (21)$$
$$D' = D - x^\mu P_\mu; \quad (22)$$
$$K'_\mu = K_\mu - 2x_\mu D + 2x^\nu M_{\mu\nu} + 2x_\mu x^\nu P_\nu - x^2 P_\mu. \quad (23)$$

As a result,

$$[M_{\mu\nu}, \varphi_\alpha(x)] = -\{\Sigma^{(\varphi)}_{\mu\nu} + i\,(x_\mu\partial_\nu - x_\nu\partial_\mu)\}^\beta_\alpha\,\varphi_\beta(x); \quad (24)$$
$$[D, \varphi_\alpha(x)] = -\{\Delta^{(\varphi)} - ix^\mu\partial_\mu\}^\beta_\alpha\,\varphi_\beta(x); \quad (25)$$
$$[K_\mu, \varphi_\alpha(x)] \\ = -\{k^{(\varphi)}_\mu + i\,(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2ix^\nu\Sigma^{(\varphi)}_{\mu\nu} + 2ix_\mu\Delta^{(\varphi)})\}^\beta_\alpha\,\varphi_\beta(x). \quad (26)$$

We shall restrict ourselves here to considering only finite-dimensional representations of the little group for which $k_\mu = 0$. This will be sufficient for a number of physical applications. There are besides these representations other finite-dimensional representations, for which $k_\mu \neq 0$, but they are nilpotent, i.e., $k^n_\mu = 0$ for some integral n, and there are also finite-dimensional representations.

From the commutation relations for $\Sigma_{\mu\nu}$, Δ, and k_μ we see that if the matrices $\Sigma_{\mu\nu}$ form an irreducible representation of the algebra of the homogeneous Lorentz group, then for $k_\mu = 0$ the matrix Δ is proportional to the unit matrix, and we can then set

$$\Delta^{(\varphi)} = il_\varphi; \quad (27)$$

l_φ is called the scale dimension of the field φ. The relations (25) and (26) now take the form

$$[D, \varphi(x)] = i\,(l_\varphi - x^\mu\partial_\mu)\,\varphi(x); \quad (28)$$
$$[K_\mu, \varphi(x)] = i\,(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2ix^\nu\Sigma^{(\varphi)}_{\mu\nu} - 2l_\varphi x_\mu)\,\varphi(x). \quad (29)$$

In particular, from (28) we obtain an expression for a finite scale transformation:

$$\exp(-i\alpha D)\,\varphi(x)\exp(i\alpha D) = \rho^{-l_\varphi}\varphi(\rho x),\quad \rho = \exp(\alpha). \quad (30)$$

Note also the relation

$$\exp(-i\alpha D)\,P^2\exp(i\alpha D) = \rho^2 P^2, \quad (31)$$

which follows from (9) and which shows that if a state A has mass m_A, then $\exp(i\alpha D)|A\rangle$ will be a state with mass ρm_A. Thus, exact scale invariance has the consequence that the eigenvalues of the mass operator are either zero or form a continuum.

2. COORDINATE INVERSION TRANSFORMATION

To study conformal invariance, it is much more convenient to use the (discrete) coordinate inversion operator

$$Rx_\mu = -x_\mu/x^2 \quad (32)$$

than the operator of the special conformal transformation (3). It is readily verified that

$$RM_{\mu\nu}R = M_{\mu\nu}; \quad (33)$$
$$RP_\mu R = K_\mu; \quad (34)$$
$$RDR = -D. \quad (35)$$

These relations show that covariance (invariance) with respect to the Poincaré algebra and the scale and R transformations together entail covariance (invariance) with respect to the conformal algebra.

We shall seek the law of transformation of the field operator under the coordinate inversion R in the general form[25]

$$U_i(R)\,\varphi(x)\,U^{-1}(R) = (x^2)^{l_\varphi}S^{(\varphi)}(x)\,\varphi(-x/x^2), \quad (36)$$

where $S(x)$ is a matrix that can be determined from the property

$$R^2 = 1 \quad (37)$$

of the R transformation and from the connection between this transformation and the transformations (33)–(35) of the conformal group. Using (37), we obtain

$$S(x)\,S(-x/x^2) = 1. \quad (38)$$

The relation (35) together with (30) gives

$$S(x)\,S(-\rho x/x^2) = 1. \quad (39)$$

Equations (38) and (39) together show that $S(x)$ is a homogeneous matrix of zeroth degree:

$$S(\rho x) = S(x),\quad \rho > 0. \quad (40)$$

Further, using (34), (18), and (26), we can derive the equation

$$(1+c^2x^2+2cx)^{l_\varphi} S(x)\, S(-x/x^2-c)\, \varphi([x+cx^2]/[1+2cx+c^2x^2]) = \varphi(x) - c^\mu \{2x_\mu x^\nu \partial_\nu - x^2\partial_\mu - 2l_\varphi x_\mu - 2ix^\nu \Sigma_{\mu\nu}\}\varphi(x) + \ldots \quad (41)$$

Comparing the terms of first order in c on both sides of the equation, we obtain

$$S(-x/x^2)\,\partial_\mu S(x) = 2i\,(x^\nu/x^2)\,\Sigma_{\mu\nu}. \quad (42)$$

This equation together with the conditions (38) and (40) enables us to determine the matrix $S(x)$ completely and, therefore, the field transformation law as well. Let us consider some specific cases.

For a spinless field, $\Sigma_{\mu\nu}=0$, and Eqs. (38), (40), and (42) give (to within a phase factor) $S(x)=1$. Thus,

$$U(R)\,\varphi(x)\,U^{-1}(R) = \delta_\varphi (x^2)^{l_\varphi}\varphi(Rx), \quad |\delta_\varphi|=1. \quad (43)$$

For a spinor field $\psi(x)$, we seek $S(x)$ in the following general form, which satisfies the conditions of Lorentz covariance and the homogeneity condition (40):

$$S(x) = a + b\hat{x}/(x^2)^{1/2}; \quad \hat{x} \equiv x_\mu \gamma^\mu, \quad (44)$$

where a and b are constants.

From Eq. (38), $a^2-b^2=1$; Eq. (42) with $\Sigma_{\mu\nu} = (i/4)[\gamma_\mu, \gamma_\nu]$ is satisfied only for $a=0$. Therefore, we have

$$U(R)\,\psi(x)\,U^{-1}(R) = \delta_\psi (x^2)^{l_\psi}[\hat{x}/(x^2)^{1/2}]\,\psi(Rx), \quad |\delta_\psi|=1. \quad (45)$$

For a vector field $V_\mu(x)$, we seek $S(x)$ in the form

$$S(x)^\nu_\mu = a\delta^\nu_\mu + bx_\mu x^\nu/x^2.$$

Equation (38) leads to the relations $a^2=1$, $2ab+b^2=0$, and Eq. (42) with $(\Sigma_{\mu\nu})^\sigma_\rho = i(g_{\mu\rho}\delta^\sigma_\nu - g_{\nu\rho}\delta^\sigma_\mu)$ gives $ab = -2$, $2a+b=0$. Therefore,

$$U(R)\,V_\mu(x)\,U^{-1}(R) = \delta_V (x^2)^{l_V}(\delta^\nu_\mu - 2x_\mu x^\nu/x^2)\,V_\nu(Rx), \quad |\delta_V|=1. \quad (46)$$

For the Rarita–Schwinger field $\psi_\mu(x)$, proceeding similarly, we obtain

$$U(R)\,\psi_\mu(x)\,U^{-1}(R) = \delta_\psi (x^2)^{l_\psi}(\delta^\nu_\mu - 2x_\mu x^\nu/x^2) \times (\hat{x}/(x^2)^{1/2})\,\psi_\nu(Rx), \quad |\delta_\psi|=1. \quad (47)$$

The generalization for a field of arbitrary spin is now obvious. For a tensor operator of nth rank we have

$$U(R)\,A_{\mu_1\ldots\mu_n}(x)\,U^{-1}(R) = \delta_A (x^2)^{l_A}(\delta^{\nu_1}_{\mu_1} - 2x_{\mu_1}x^{\nu_1}/x^2) \ldots (\delta^{\nu_n}_{\mu_n} - 2x_{\mu_n}x^{\nu_n}/x^2)\,A_{\nu_1\ldots\nu_n}(Rx), \quad |\delta_A|=1. \quad (48)$$

3. COVARIANT TWO-POINT AND THREE-POINT FUNCTIONS

As already mentioned in the Introduction, the requirement of conformal invariance has made it possible to determine almost uniquely the two- and three-point correlation functions. In this section, on the basis of the R transformation law, we derive general expressions for these functions.

We consider first the two point function of symmetric and traceless tensor operators $A_{\mu_1\cdots\mu_p}(x)$ and $B_{\nu_1\cdots\nu_q}(y)$ with dimensions l_A and l_B, respectively. We denote

$$G^{AB}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}(x, y) \equiv \langle 0|\,A_{\mu_1\ldots\mu_p}(x)\,B_{\nu_1\ldots\nu_q}(y)\,|0\rangle. \quad (49)$$

The condition of scale invariance requires that [see (1.30)]

$$G^{AB}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}(\lambda x, \lambda y) = \lambda^{l_A+l_B}\bar{G}^{AB}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}(x, y), \quad (50)$$

and the condition of R invariance requires that

$$G^{AB}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}(Rx, Ry) = (x^2)^{-l_A}(y^2)^{-l_B} \times M(x)^{\mu_1'}_{\mu_1}\ldots M(x)^{\mu_p'}_{\mu_p} M(y)^{\nu_1'}_{\nu_1}\ldots M(y)^{\nu_q'}_{\nu_q} G^{AB}_{\mu_1'\ldots\mu_p'\nu_1'\ldots\nu_q'}(x, y) \quad (51)$$

[see (48); the phase factors are omitted].

Here, we have used the notation

$$M(x)^{\mu'}_\mu \equiv \delta^{\mu'}_\mu - 2x_\mu x^{\mu'}/x^2. \quad (52)$$

To find the general form of the solution of Eqs. (50) and (51), we begin with the identities

$$(Rx-Ry)^2 = (x-y)^2/x^2 - y^2; \quad (53)$$

$$R(Rx-Ry)_\mu = -x^2 M(x)^{\mu'}_\mu R(x-y)_{\mu'} + x_\mu; \quad (54)$$

$$M(x)^{\mu_1'}_{\mu_1} M(x)^{\mu_2'}_{\mu_2} g_{\mu_1'\mu_2'} = g_{\mu_1\mu_2}; \quad (55)$$

$$M(x)^{\mu'}_\mu M(y)^{\nu'}_\nu M_{\mu'\nu'}(x-y) = M_{\mu\nu}(Rx-Ry), \quad (56)$$

which follow directly from the definitions (32) and (52). Finally, we have

$$G^{AB}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q}(x, y) = \delta_{pq}\delta_{l_A l_B} G^{AB}[(x-y)^2]^{l_A} \times \sum_k (-1)^k k!\,\frac{(p-k)!}{p!}\,\underset{(\mu)}{S}\,\underset{(\nu)}{S}\, g_{\mu_1\mu_2}\ldots g_{\mu_{2k-1}\mu_{2k}} \times g_{\nu_1\nu_2}\ldots g_{\nu_{2k-1}\nu_{2k}} M(x-y)_{\mu_{2k+1}\nu_{2k+1}}\ldots M(x-y)_{\mu_p\nu_q}, \quad (57)$$

where k takes integral values from zero to $p/2$; $S_{(\mu)}$ and $S_{(\nu)}$ denote symmetrization with respect to all indices μ and ν separately. It can be seen that the number of terms which occur here is $[p!(2k-1)!!/(2k)!]^2 \times [1/(p-2k)!]$. It is also easy to show[21] that if $A_{\mu_1\cdots\mu_p}$ or $B_{\nu_1\cdots\nu_q}$ is a conserved operator, i.e., if

$$\partial^\mu A_{\mu\mu_2\ldots\mu_p} = 0 \quad \text{or} \quad \partial^\nu B_{\nu\nu_2\ldots\nu_q} = 0,$$

then this function can be nonzero only for $l_A = -(p+2)$.

In the simplest case when A and B are scalar operators,

$$G^{AB}(x, y) = \langle 0|\,A(x)\,B(y)\,|0\rangle = \delta_{l_A l_B} G^{AB}[(x-y)^2]^{l_A}. \quad (58)$$

The two-point function of spinor fields:

$$\{G^{\psi\chi}(x-y)\}^\beta_\alpha \equiv \langle 0|\,\psi_\alpha(x)\,\bar{\chi}^\beta(y)\,|0\rangle \quad (59)$$

satisfies the equations

$$G^{\psi\chi}(\lambda x, \lambda y) = \lambda^{l_\psi+l_\chi} G^{\psi\chi}(x, y), \quad (60)$$

$$G^{\psi\chi}(Rx, Ry) = (x^2)^{-l_\psi}(y^2)^{-l_\chi}[\hat{x}/(x^2)^{1/2}]\,G^{\psi\chi}(x, y)\,[\hat{y}/(y^2)^{1/2}] \quad (61)$$

and therefore has the form

$$G^{\psi\chi}(x, y) = \delta_{l_\psi l_\chi} iG^{\psi\chi}[(x-y)^2]^{l_\psi}(\hat{x}-\hat{y})/[(x-y)^2]^{1/2}. \quad (62)$$

We have used the identity

$$\hat{x}\widehat{R(x-y)}\,\hat{y} \equiv \widehat{R(Rx-Ry)}. \quad (63)$$

We now consider the three-point function

$$G^{ABC}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q\rho_1\ldots\rho_r}(x, y, z) \equiv \langle 0|\,A_{\mu_1\ldots\mu_p}(x)\,B_{\nu_1\ldots\nu_q}(y)\,C^{(z)}_{\rho_1\ldots\rho_r}\,|0\rangle. \quad (64)$$

It satisfies the equations

$$G^{ABC}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q\rho_1\ldots\rho_r}(\lambda x, \lambda y, \lambda z) = \lambda^{l_A+l_B+l_C} G^{ABC}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q\rho_1\ldots\rho_r}(x, y, z); \quad (65)$$

$$G^{ABC}_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q\rho_1\ldots\rho_r}(Rx, Ry, Rz) = (x^2)^{-l_A}(y^2)^{-l_B}(z^2)^{-l_C} M(x)^{\mu_1'}_{\mu_1}\ldots M(x)^{\mu_p'}_{\mu_p} \times M(y)^{\nu_1'}_{\nu_1}\ldots M(y)^{\nu_q'}_{\nu_q} M(z)^{\rho_1'}_{\rho_1}\ldots M(z)^{\rho_r'}_{\rho_r} G^{ABC}_{\mu_1'\ldots\mu_p'\nu_1'\ldots\nu_q'\rho_1'\ldots\rho_r'}. \quad (66)$$

Using again the identities (53)–(56), we can find the general expression for the three-point function $G^{ABC}_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q\rho_1\cdots\rho_r}(x, y, z)$ of all irreducible symmetric traceless tensor operators.[21] In particular, for scalar and conserved irreducible tensor operators only three-point functions of the following types can have a nonvanishing value:

$$\langle 0 | A(x) B(y) C(z) | 0 \rangle = G^{ABC} [(x-y)^2]^{(l_A+l_B-l_C)/2} [(y-z)^2]^{(l_B+l_C-l_A)/2} [(z-x)^2]^{(l_C+l_A-l_B)/2}; \quad (67)$$

$$\langle 0 | A_\mu(x) B_\nu(y) C(z) | 0 \rangle = G^{ABC} [(x-y)^2]^{-1/2} [(y-z)^2]^{-3/2} [(z-x)^2]^{-3/2} [R(x-y) - R(x-z)]_\mu [R(y-z) - R(y-x)]_\nu \quad (68)$$

for $l_A = l_B = l_C = -3$;

$$\langle 0 | A_\mu(x) B_\nu(y) C_\rho(z) | 0 \rangle = G^{ABC} [(x-y)^2 (y-z)^2 (z-x)^2]^{-1} [R(x-y) - R(x-z)]_\mu [R(y-z) - R(y-x)]_\nu [R(z-x) - R(z-y)]_\rho \quad (69)$$

for $l_A = l_B = l_C = -3$;

$$\langle 0 | A_{\mu_1 \ldots \mu_p}(x) B(y) C(z) | 0 \rangle = G^{ABC} \sum_k \frac{(-1)^k}{2^k} \cdot \frac{(p-k)!}{p!} [(x-y)^2]^{-(1+k)} [(y-z)^2]^{l_B+1+k} \times [(z-x)^2]^{-(1+k)} \underset{(\mu)}{S} g_{\mu_1\mu_2} \ldots g_{\mu_{2k-1}\mu_{2k}} [R(x-y) - R(x-z)]_{\mu_{2k+1}} \ldots [R(x-y) - R(x-z)]_{\mu_p} \quad (70)$$

for $l_A = -(p+2)$, $l_B = l_C$. Here, k takes integral values from zero to $p/2$, and the number of terms in $S_{(\mu)}$ is $p![(2k)!(p-2k)!]^{-1}(2k-1)!!$.

Similarly, one can find a three-point function that contains spinor operators:

$$\{G^{ABC}_{\rho_1 \ldots \rho_r}(x, y, z)\}^\beta_\alpha \equiv \langle 0 | A_\alpha(x) \overline{B}^\beta(y) C_{\rho_1 \ldots \rho_r}(z) | 0 \rangle. \quad (71)$$

In particular, we have

$$\langle 0 | \psi(x) \overline{\psi}(y) J_\mu(z) | 0 \rangle = [(x-y)^2]^{-(2l_\psi+3)/2} \times [(y-z)^2]^{-1} [(z-x)^2]^{-1} \Big\{ [R(z-x) - R(z-y)]_\mu \times \Big[g \frac{\hat{x}-\hat{y}}{(x-y)^2} + g' \frac{(\hat{x}-\hat{z})(\hat{y}-\hat{z})}{((x-y)^2 (y-z)^2 (z-x)^2)^{1/2}} \Big] + g'' \frac{\hat{x}-\hat{z}}{(x-z)^2} \gamma_\mu \frac{\hat{y}-\hat{z}}{(y-z)^2} \Big\} \quad (72)$$

for the conserved vector current J_μ and $l_{J_\mu} = -3$ and

$$\langle 0 | \psi(x) \overline{\psi}(y) \phi(z) | 0 \rangle = [(x-y)^2]^{(2l_\psi - l_\phi + 1)/2} [(y-z)^2]^{l_\phi/2} [(z-x)^2]^{l_\phi/2} \times \Big\{ g \frac{\hat{x}-\hat{y}}{(x-y)^2} + g' \frac{(\hat{x}-\hat{z})(\hat{y}-\hat{z})}{((x-y)^2 (y-z)^2 (z-x)^2)^{1/2}} \Big\} \quad (73)$$

for the scalar operator ϕ.

4. DIMENSIONAL PROPERTIES OF THE ENERGY-MOMENTUM TENSOR

For every given Lagrangian there exists the canonical energy-momentum tensor

$$T^c_{\rho\mu} = \sum_\varphi \pi_\rho \partial_\mu \varphi - g_{\rho\mu} \mathcal{L}, \quad (74)$$

where

$$\pi_\rho \equiv \partial \mathcal{L} / \partial (\partial^\rho \varphi),$$

by means of which one can form the generators of Poincaré transformations using the canonical equal-time commutation relations:

$$P_\mu = \int d\mathbf{x} T^c_{0\mu}; \quad (75)$$

$$M_{\mu\nu} = \int d\mathbf{x} \mathcal{M}^c_{0,\,\mu\nu}; \quad (76)$$

where

$$\mathcal{M}^c_{\rho;\,\mu\nu} \equiv x_\mu T^c_{\rho\nu} - x_\nu T^c_{\rho\mu} - i \sum_\varphi \pi_\rho \Sigma_{\mu\nu} \varphi \quad (77)$$

is the angular momentum tensor.

In the framework of the canonical formalism, we also find the scale and conformal currents whose spatial integrals of the time components realize the generators of the scale and conformal transformations, respectively. They are[13]

$$\mathcal{D}^c_\mu = \sum_\varphi l_\varphi \pi_\mu \varphi - x^\nu T^c_{\mu\nu}; \quad (78)$$

$$\mathcal{K}^c_{\rho\mu} = 2x_\mu x^\nu T^c_{\rho\nu} - x^2 T^c_{\rho\mu} - \sum_\varphi \pi_\rho (2ix^\nu \Sigma_{\mu\nu} + 2l_\varphi x_\mu) \varphi. \quad (79)$$

We go over from the canonical energy-momentum tensor to the symmetric Belifante tensor, which has the form

$$T^B_{\rho\mu} = T^c_{\rho\mu} + \partial^\sigma f_{\mu;\,\rho\sigma}; \qquad f_{\mu;\,\rho\sigma} \equiv \frac{i}{2} \sum_\varphi \{\pi_\sigma \Sigma_{\mu\rho} \varphi - \pi_\rho \Sigma_{\mu\sigma} \varphi - \pi_\mu \Sigma_{\rho\sigma} \varphi\}. \quad (80)$$

The Belifante tensor $T^B_{\rho\mu}$ has the same conservation property as $T^c_{\rho\mu}$, and the spatial integral of $T^B_{0\mu}$ is equal to the spatial integral of $T^c_{0\mu}$ and can therefore also serve as translation generator. In addition, the tensor

$$\mathcal{M}^B_{\rho;\,\mu\nu} \equiv x_\mu T^B_{\rho\nu} - x_\nu T^B_{\rho\mu} \quad (81)$$

formed from it has all the properties inherent in the angular momentum tensor.

Further, as was shown by Callan, Coleman, and Jackiw,[26] the energy-momentum tensor can be redefined in such a way that the scale and conformal currents can be readily expressed in terms of it. A sufficient condition for this is that $\sum_\varphi (l_\varphi \pi_\mu \varphi + i\pi^\nu_{\mu\nu} \varphi)$ be expressible as the divergence of some second-rank tensor $A_{\mu\nu}$:

$$\sum_\varphi (l_\varphi \pi_\mu \varphi + i\pi^\nu \Sigma_{\mu\nu} \varphi) = -\partial^\nu A_{\mu\nu}. \quad (82)$$

It was found that[26]

$$\theta_{\mu\nu} = T^B_{\mu\nu} + \partial^\lambda \partial^\rho X_{\lambda\rho\mu\nu}/2, \quad (83)$$

where

$$X_{\lambda\rho\mu\nu} \equiv g_{\lambda\rho} A^+_{\mu\nu} - g_{\lambda\mu} A^+_{\rho\nu} - g_{\lambda\nu} A^+_{\mu\rho} + g_{\mu\nu} A^+_{\lambda\rho} - (g_{\mu\nu} g_{\lambda\rho} + g_{\lambda\mu} g_{\rho\nu}) A^{+\sigma}_\sigma/3; \quad (84)$$

$A^+_{\mu\nu}$ is the symmetric part of the tensor $A_{\mu\nu}$:

$$A^+_{\mu\nu} \equiv (A_{\mu\nu} + A_{\nu\mu})/2. \quad (85)$$

This new tensor $\theta_{\mu\nu}$ is called the improved energy-momentum tensor.

It is readily seen that the expressions

$$\mathcal{D}_\rho = -x^\nu \theta_{\mu\nu}; \quad (86)$$

$$\mathcal{K}_{\rho\mu} \equiv (2x_\mu x_\nu - g_{\mu\nu} x^2) \theta^\nu_\rho \quad (87)$$

have the properties

$$\int d\mathbf{x} \mathcal{D}_0(x) = \int d\mathbf{x} \mathcal{D}^c_0(x); \quad (88)$$

$$\int d\mathbf{x} \mathcal{K}_{0\mu}(x) = \int d\mathbf{x} \mathcal{K}^c_{0\mu}(x); \quad (89)$$

$$\partial^\mu \mathcal{D}_\mu = \partial^\mu \mathcal{D}^c_\mu = -\theta^\mu_\mu; \quad (90)$$

$$\partial^\rho \mathcal{K}_{\rho\mu} = \partial^\rho \mathcal{K}^c_{\rho\mu} = 2x_\mu \theta^\rho_\rho \quad (91)$$

and can therefore serve as scale and conformal currents. In addition, their divergences are proportional to the trace of the improved energy-momentum tensor θ^μ_μ, so that the nature of the breaking of the scale and conformal invariance is determined by the behavior of this tensor. The hope that scale (conformal) symmetry is a useful approximate symmetry takes the form that θ^μ_μ should be a "small" operator whose matrix elements vanish at high energies.

As regards the angular momentum tensor, we can express it in a form analogous to

$$\mathcal{M}_{\rho;\mu\nu}=x_\mu\theta_{\rho\nu}-x_\nu\theta_{\rho\mu}. \quad (92)$$

The condition (92) holds in many models, and it is usually assumed that it is satisfied. It is also assumed that for the theory of gravitation the energy-momentum tensor is to be chosen such that the scale and conformal currents be expressible in terms of it in accordance with Eqs. (86) and (87) and the angular momentum tensor similarly in terms of (92).

Note that the energy-momentum tensor $\theta_{\mu\nu}$ can be called improved in a further aspect—its matrix elements are finite in any order of renormalized perturbation theory,[26] i.e., they do not depend on the cutoff in the limit of large cutoff.

We give one further expression for the matrix elements of $\theta_{\mu\nu}$ that will be needed in what follows. We shall use the normalization

$$\langle A(\mathbf{p}',\alpha')\,|\,A(\mathbf{p},\alpha)\rangle=(2\pi)^3(p_0/\lambda_A)\,\delta(\mathbf{p}'-\mathbf{p})\,\delta_{\alpha\alpha'}, \quad (93)$$

where λ_A is equal to 1/2 if A is a boson particle, and m_A if A is a fermion particle; α and α' are other quantum numbers in addition to the momentum. It follows from this normalization that

$$\langle A(\mathbf{p},\alpha')\,|\,\theta_{0\nu}(0)\,|\,A(\mathbf{p},\alpha)\rangle=\frac{p_0p_\alpha}{\lambda_A}\,\delta_{\alpha\alpha'}. \quad (94)$$

Differentiating with respect to p'_k the equation

$$(p'-p)^\mu\langle A(\mathbf{p},\alpha')\,|\,\theta_{\mu\nu}(0)\,|\,A(\mathbf{p},\alpha)\rangle=0, \quad (95)$$

which follows from the condition of conservation of the tensor $\theta_{\mu\nu}$, and setting $\mathbf{p}'=\mathbf{p}$, we obtain, using (94),

$$\langle A(\mathbf{p},\alpha')\,|\,\theta_{\mu\nu}(0)\,|\,A(\mathbf{p},\alpha)\rangle=\frac{p_\mu p_\nu}{\lambda_A}\,\delta_{\alpha\alpha'}. \quad (96)$$

This shows, in particular, that the matrix elements of the components of $\theta_{\mu i}$ for the state of any particle at rest are zero. In addition,

$$\langle A(\mathbf{p},\alpha')\,|\,\theta^\mu_\mu(0)\,|\,A(\mathbf{p},\alpha)\rangle=\delta_{\alpha\alpha'}\begin{cases}2m_A^2, & \text{for } A \text{ a boson;}\\ m_A, & \text{for } A \text{ a fermion}.\end{cases} \quad (97)$$

Because particles have masses, the Hamiltonian density cannot have a definite scale invariance, and in the best case one can expect the Hamiltonian to consist of several parts with different dimensions. The determination of these dimensions is of great importance, especially when one is studying the consequences of broken scale and chiral invariance considered together (see Sec. 7).

If a component of $\theta_{\mu\nu}$ has a definite scale dimension $l_{(\mu\nu)}$, this means that the following commutation relation holds:

$$[D(x_0),\,\theta_{\mu\nu}(x)]=-\mathrm{i}\,(l_{(\mu\nu)}-x^\rho\partial_\rho)\,\theta_{\mu\nu}. \quad (98)$$

On the other hand, it can be seen from (86) that the commutator on the left-hand side of (98) can be calculated if the equal-time commutators $[\theta_{0\rho}(x),\theta_{\mu\nu}(y)]$ are determined up to the Schwinger terms of first order.

From general considerations, we can write

$$[\theta_{\mu\nu}(x),\,\theta_{\sigma\rho}(y)]_{x_0=y_0}=\sum_{r=0}^{n}F_{\sigma\rho;\,\mu\nu,\,k_1\ldots k_r}(y)\,\partial^{k_1}\ldots\partial^{k_r}\delta(\mathbf{x}-\mathbf{y}), \quad (99)$$

where $F_{\sigma\rho;\,\mu\nu,k_1\cdots k_r}$ are certain operator structure functions. The terms on the right-hand side of (99) corresponding to the values $r=1,2,\ldots$, are called Schwinger terms of first, second, etc., order. Using the laws of Lorentz transformations, we can obtain relations for these structure functions, which lead to the relations[27]

$$[\theta_{00}(x),\,\theta_{00}(y)]=-\mathrm{i}\partial_0\theta_{00}(y)\,\delta(\mathbf{x}-\mathbf{y})-2\mathrm{i}\theta_{0k}(y)\,\partial^k\delta(\mathbf{x}-\mathbf{y})+\ldots; \quad (100)$$

$$[\theta_{00}(x),\,\theta_{0i}(y)]=-\mathrm{i}\partial_0\theta_{0i}(y)\,\delta(\mathbf{x}-\mathbf{y})-\mathrm{i}\theta_{ik}(y)\,\partial^k\delta(\mathbf{x}-\mathbf{y})+\mathrm{i}\theta_{00}(y)\,\partial_i\delta(\mathbf{x}-\mathbf{y})+\ldots; \quad (101)$$

$$[\theta_{0i}(x),\,\theta_{00}(y)]=-\mathrm{i}\partial_i\theta_{00}(y)\,\delta(\mathbf{x}-\mathbf{y})-\mathrm{i}\theta_{ik}(y)\,\partial^k\delta(\mathbf{x}-\mathbf{y})+\mathrm{i}\theta_{00}(y)\,\partial_i\delta(\mathbf{x}-\mathbf{y})+\ldots; \quad (102)$$

$$[\theta_{0i}(x),\,\theta_{0j}(y)]=-\mathrm{i}\partial_i\theta_{0j}(y)\,\delta(\mathbf{x}-\mathbf{y})+\mathrm{i}\theta_{0j}(y)\,\partial_i\delta(\mathbf{x}-\mathbf{y})+\mathrm{i}\theta_{0i}(y)\,\partial_j\delta(\mathbf{x}-\mathbf{y})+\ldots, \quad (103)$$

where $x_0=y_0$; $i,j,\ldots=1,2,3,\ldots$; and the ellipsis denotes Schwinger terms of second and higher orders.

To find $[\theta_{0\mu}(x),\theta_{jk}(y)]$ we proceed similarly. We proceed from the commutators (100)–(103), using the Jacobi identities written for the three operators $\int d\mathbf{x}\,x^k\theta_{0\nu}(0,\mathbf{x})$, $\theta_{0i}(0)$ and M_{0j}.

It follows from (100)–(103) that

$$[\mathcal{D}(x_0),\,\theta_{0\mu}(x)]=-\mathrm{i}\,(-4-x^\nu\partial_\nu)\,\theta_{0\mu}(x)-\mathrm{i}g_{0\mu}\theta^\nu_\nu(x). \quad (104)$$

This equation shows that θ_{00} does not have a definite scale dimension. If it is represented in the form

$$\theta_{00}(x)=\overline{\theta}_{00}(x)+\sum_i u_i(x)+\mathscr{C}(x), \quad (105)$$

where $\overline{\theta}_{00}$ has dimension four, u_i has scale dimension l_i, and $\mathscr{C}$ is some c number, then

$$[\mathcal{D}(x_0),\,\theta_{00}(x)]=-\mathrm{i}\,(-4-x^\mu\partial_\mu)\,\theta_{00}+\mathrm{i}\sum_i(4+l_i)\,u_i-4\mathrm{i}\mathscr{C}(x)-\mathrm{i}x^\mu\partial_\mu\mathscr{C}(x). \quad (106)$$

It follows from (104) and (106) that

$$\theta^\mu_\mu(x)=\sum_i(4+l_i)\,u_i(x)+4\mathscr{C}(x)+x^\mu\partial_\mu\mathscr{C}(x), \quad (107)$$

which at the point $x=0$ gives

$$\theta^\mu_\mu(0)=\sum_i(4+l_i)\,u_i(0)+4\mathscr{C}(0). \quad (108)$$

This is the theorem of Ref. 28.

From the expressions (86), (87), and (92) for the conformal currents and from the commutation relations (100)–(103) we obtain by direct calculation the commutators of the corresponding charges:

$$[\mathcal{M}_{\mu\nu}\mathscr{D}(x_0)]=-\mathrm{i}\int d\mathbf{x}\,(g_{\mu0}x_\nu-g_{\nu0}x_\mu)\,\theta^\sigma_\sigma(x); \quad (109)$$

$$[\mathcal{M}_{\mu\nu},\,\mathscr{K}_\rho(x_0)]=\mathrm{i}\,(g_{\nu\rho}\mathscr{K}_\mu-g_{\mu\rho}\mathscr{K}_\nu)+2\mathrm{i}\int d\mathbf{x}\,(g_{\mu0}x_\nu-g_{\nu0}x_\mu)\,x_\rho\theta^\sigma_\sigma; \quad (110)$$

$$[\mathscr{P}_\mu,\,\mathscr{D}(x_0)]=-\mathrm{i}\,\mathscr{P}_\mu+\mathrm{i}g_{\mu0}\int d\mathbf{x}\,\theta^\sigma_\sigma(x); \quad (111)$$

$$[\mathscr{P}_\mu,\,\mathscr{K}_\nu(x_0)]=-2\mathrm{i}\,(g_{\mu\nu}\mathscr{D}(x_0)+\mathcal{M}_{\mu\nu})-2\mathrm{i}g_{\mu0}x_0\int d\mathbf{x}\,\theta^\sigma_\sigma(x); \quad (112)$$

$$[\mathscr{D}(x_0),\,\mathscr{K}_\mu(x_0)]=-\mathrm{i}\mathscr{K}_\mu(x_0)+\mathrm{i}g_{\mu0}\int d\mathbf{x}\,x^2\theta^\sigma_\sigma(x); \quad (113)$$

$$[\mathscr{K}_\mu(x_0),\,\mathscr{K}_\nu(x_0)]=2\mathrm{i}\int d\mathbf{x}\,(x_\nu g_{\mu0}-x_\mu g_{\nu0})\,\theta^\sigma_\sigma(x). \quad (114)$$

These relations are identical with the original relations (7)–(12) only in the limit $\theta^\sigma_\sigma\to0$, when $\mathscr{D}_\mu(x)$ and $\mathscr{K}_{\rho\mu}(x)$ can be assumed to be conserved, as one would expect.

5. MASS FORM FACTOR AND MASS RADIUS OF THE PION

If it is assumed that the energy-momentum tensor $\theta_{\mu\nu}$ is the source of the gravitational field, one can draw conclusions about the distribution of mass within a particle by studying the matrix elements between its states. The resulting form factors are called the mass (or gravitational) form factors.

In the framework of the theory of conformal invariance, using the properties of the energy-momentum tensor, one can estimate the pion mass radius, which is proportional to the derivative of the corresponding mass form factor at $t=0$. For this, we use a method based on the conformal Ward identities and PCAC. The latter is used to derive a low energy theorem for a matrix element with soft pion, by analogy with the current-algebra method.

We consider the chronological product of the scale and the conformal currents and the fields $\varphi_i(x)$: $T\{\mathscr{D}_\rho(x)\varphi_1(x_1)\cdots\varphi_n(x_n)\}$ and $T\{\mathscr{K}_{\rho\mu}(x)\varphi_1(x_1)\cdots\varphi_n(x_n)\}$. On the basis of the field transformation laws (28) and (29), using the standard technique, from the expressions (86) and (87) we obtain[29]

$$\tau^{(\varphi_1\ldots\varphi_n\theta)}(p_1,\ldots,p_n;0)=-\mathrm{i}\sum_{j=1}^{n}\left(l_j+4+p_j^\nu\frac{\partial}{\partial p_j^\nu}\right)\tau^{(\varphi_1\ldots\varphi_n)}(p_1,\ldots,p_n); \tag{115}$$

$$\frac{\partial\tau^{(\varphi_1\ldots\varphi_n\theta)}}{\partial k^\mu}(p_1,\ldots,p_n;k)\Big|_{k=0}=-\mathrm{i}\sum_{j=1}^{n}\Big\{(4+l_j)\frac{\partial}{\partial p_j^\mu}+p_j^\nu\frac{\partial^2}{\partial p_j^\nu\partial p_j^\mu}-\frac{1}{2}p_{j\mu}\Box(p_j)+\mathrm{i}\Sigma_{\mu\nu}^{\varphi_j}\frac{\partial}{\partial p_{j\nu}}\Big\}\tau^{(\varphi_1\ldots\varphi_n)}(p_1,\ldots p_n), \tag{116}$$

where τ denotes the Fourier transforms of the vacuum expectation values of the corresponding chronological products:

$$\tau^{(\varphi_1\ldots\varphi_n)}(p_1,\ldots,p_n)=\int d^4x_1\ldots d^4x_n\times\exp\left[\mathrm{i}\sum_{p=1}^{n}p_jx_j\right]\langle0|T\{\varphi_1(x_1)\ldots\varphi_n(x_n)\}|0\rangle; \tag{117}$$

$$\tau^{(\varphi_1\ldots\varphi_n\theta)}(p_1,\ldots,p_n;k)=\int d^4x_1\ldots d^4x_n\,d^4x\exp\left[\mathrm{i}\left(\sum_j p_jx_j+kx\right)\right]\times\langle0|T\{\varphi_1(x_1)\ldots\varphi_n(x_n)\,\theta_\mu^\mu(x)\}|0\rangle. \tag{118}$$

Using the translational invariance, we can transform (115) and (116) to the more convenient form

$$\Gamma^{(\varphi_1\ldots\varphi_n\theta)}\left(p_1,\ldots,p_{n-1},-\sum_{j=1}^{n-1}p_j\right)=-\mathrm{i}\left[l_{\varphi_n}+4(n-1)+\sum_{j=1}^{n-1}\left(l_{\varphi_j}+p_j^\nu\frac{\partial}{\partial p_j^\nu}\right)\right]\times\Gamma^{(\varphi_1\ldots\varphi_n)}(p_1,\ldots,p_{n-1}); \tag{119}$$

$$\frac{\partial\Gamma^{(\varphi_1\ldots\varphi_n\theta)}}{\partial k^\mu}\left(p_1,\ldots,p_{n-1},-\sum_{j=1}^{n-1}p_j-k\right)\Big|_{k=0}=-\mathrm{i}\sum_{j=1}^{n-1}\Big\{(4+l_{\varphi_j})\frac{\partial}{\partial p_j^\mu}+p_j^\nu\frac{\partial}{\partial p_j^\nu\partial p_j^\mu}-\frac{1}{2}p_{j\mu}\Box(p_j)+\mathrm{i}\Sigma_{\mu\nu}^{(\varphi_j)}\frac{\partial}{\partial p_{j\nu}}\Big\}\Gamma^{(\varphi_1\ldots\varphi_n)}(p_1,\ldots,p_{n-1}), \tag{120}$$

where we have introduced the notation

$$\Gamma^{(\varphi_1\ldots\varphi_n)}(p_1,\ldots,p_{n-1})\equiv\int d^4x_1\ldots d^4x_{n-1}\times\exp\left[\mathrm{i}\sum_{j=1}^{n-1}p_jx_j\right]\langle0|T\{\varphi_1(x_1)\ldots\varphi_{n-1}(x_{n-1})\varphi_n(0)\}|0\rangle; \tag{121}$$

$$\Gamma^{(\varphi_1\ldots\varphi_n\theta)}(p_1,\ldots,p_n)\equiv\int d^4x_1\ldots d^4x_n\exp\left[\mathrm{i}\sum_{j=1}^{n}p_jx_j\right]\times\langle0|T\{\varphi_1(x_1)\ldots\varphi_n(x_n)\,\theta_\mu^\mu(0)\}|0\rangle. \tag{122}$$

In particular, for $n=2$ and $\varphi_1=\varphi_2^+\equiv\varphi$ Eqs. (119)–(123) make it possible to relate the $\theta\varphi\varphi^+$ vertex to the corresponding Feynman propagator

$$\widetilde{\Delta}_F(p)\equiv\int d^4x\exp(\mathrm{i}px)\langle0|T\{\varphi(x)\varphi^+(0)\}|0\rangle \tag{123}$$

and obtain

$$\Gamma^{(\varphi\varphi^+\theta)}(p-p)=-\mathrm{i}\left[2l_\varphi+4+p^\nu\frac{\partial}{\partial p^\nu}\right]\widetilde{\Delta}_F(p); \tag{124}$$

$$\frac{\partial\Gamma^{(\varphi\varphi^+\theta)}}{\partial p'^\mu}(p,-p')\Big|_{p'=p}=\mathrm{i}\Big\{(4+l_\varphi)\frac{\partial}{\partial p^\mu}+p^\nu\frac{\partial}{\partial p^\nu p^\mu}-\frac{1}{2}p_\mu\Box(p)+\mathrm{i}\Sigma_{\mu\nu}^{(\varphi)}\frac{\partial}{\partial p_\nu}\Big\}\widetilde{\Delta}_F(p). \tag{125}$$

These expressions will be used to study the mass form factor of the pion, to which we now turn.

The pion mass form factor is defined as the matrix element of θ_μ^μ between the single-pion states:

$$\langle\pi^0(p)|\theta_\mu^\mu(0)|\pi^0(p')\rangle\equiv\theta(p^2,p'^2;t),\quad t\equiv(p'-p)^2. \tag{126}$$

We consider first the (noncovariant) Green's function

$$T(p^2,p'^2;t)\equiv-(p^2-m_\pi^2)(p'^2-m_\pi)\int d_x^4d_y^4\exp[\mathrm{i}(px-p'y)]\times\langle0|T\{\varphi(x)\varphi(y)\theta_\mu^\mu(0)\}|0\rangle, \tag{127}$$

where ϕ is the field operator of the π^0 meson. The matrix element (126) is obtained from (127) by replacing the noncovariant chronological product by the covariant T^* function, which is constructed by the method set forth in Ref. 30. For this, we need the equal-time commutation relations between $\theta_{\mu\nu}$ and ϕ. It can be shown that[29]

$$T^*\{\varphi(x)\varphi(y)\theta_\mu^\mu(z)\}=T\{\varphi(x)\varphi(y)\theta_\mu^\mu(z)\}+\mathrm{i}l_\varphi T\{\varphi(x)\varphi(y)\}[\delta^{(4)}(z-x)+\delta^{(4)}(z-y)]. \tag{128}$$

Thus, $\theta(p^2,p'^2;t)$ and $T(p^2,p'^2;t)$ are related by

$$\theta(p^2,p'^2;t)=T(p^2,p'^2;t)-\mathrm{i}l_\varphi(p^2-m_\pi^2)(p'^2-m_\pi)[\widetilde{\Delta}_F(p)+\widetilde{\Delta}_F(p')]. \tag{129}$$

It follows, in particular, that on the mass shell

$$\theta(m_\pi^2,m_\pi^2;t)=T(m_\pi^2,m_\pi^2;t). \tag{130}$$

The identity (124), applied to the Green's function (127), gives

$$T(p^2,p'^2;0)=\mathrm{i}(p^2-m_\pi^2)^2\left[2l_\varphi+4+p^\nu\frac{\partial}{\partial p^\nu}\right]\widetilde{\Delta}_F(p). \tag{131}$$

Using the Källén–Lehmann representation for $\widetilde{\Delta}_F(p)$:

$$\widetilde{\Delta}_F(p)=\mathrm{i}\int da^2\rho(a^2)/(p^2-a^2+\mathrm{i}\varepsilon);\quad\rho(a^2)=\delta(a^2-m_\pi^2)+\sigma(a^2);\quad\sigma(a^2)\equiv(2\pi)^3\sum_{m\neq n}\delta^{(4)}(p_n-a)|\langle0|\varphi(0)|n\rangle|^2, \tag{132}$$

we rewrite (131) in the form

$$T(p^2,p^2;0)=2(p^2-m_\pi^2)^2\Big\{\frac{m_\pi^2-(l_\varphi+1)(p^2-m_\pi^2)}{(p^2-m_\pi^2-\mathrm{i}\varepsilon)^2}+\int_{(3m_\pi)^2}^{\infty}da^2\frac{\sigma(a^2)}{(p^2-a^2-\mathrm{i}\varepsilon)^2}[a^2-(l_\varphi+1)(p^2-a^2)]\Big\}. \tag{133}$$

As $p^2\to m_\pi^2$ and $p^2\to0$, we find from this

$$T(m_\pi^2,m_\pi^2;0)=2m_\pi^2; \tag{134}$$

$$T(0,0;0)=2m_\pi^2(l_\varphi+2)\left[1+m_\pi^2\int_{(3m_\pi)^2}^{\infty}da^2\frac{\sigma(a^2)}{a^2}\right]. \tag{135}$$

Note that the result (134) can also be obtained directly from (130) since on the mass shell [see (97)]

$$\langle \pi (p) \,|\, 0^{\mu}_{\mu} \,|\, \pi (p)\rangle = 2m^2_{\pi}.$$

We now apply the identity (125). From (122) and (127),

$$\frac{\partial \Gamma^{(\varphi\varphi 0)}}{\partial p'^{\mu}} (p,\ -p')\Big|_{p'=p}$$

$$= \frac{2p_{\mu}}{(p^2 - m^2_{\pi})^2}\, T(p^2,\ p^2;\ 0) - \frac{2p_{\mu}}{(p^2 - m^2_{\pi})^2}\, \frac{\partial T(p^2,\ p'^2;\ 0)}{\partial p'^2}\Big|_{p'=p}. \qquad (136)$$

Equations (125), (131), and (136) together give

$$2p_{\mu} \frac{\partial T}{\partial p'^2} (p^2,\ p'^2;\ 0)\Big|_{p'=p}$$

$$= \mathrm{i}\,(p^2 - m^2_{\pi}) \left\{ (p^2 - m^2_{\pi}) \left[(4 + l_{\varphi}) \frac{\partial}{\partial p^{\mu}} + p^{\nu} \frac{\partial^2}{\partial p^{\nu}\, \partial p^{\mu}} - \frac{1}{2}\, p_{\mu} \Box\, (p) \right] \right.$$

$$\left. + 2p_{\mu} \left(2l_{\varphi} + 4 + p^{\nu} \frac{\partial}{\partial p^{\nu}} \right) \widetilde{\Delta}_F\, (p). \right. \qquad (137)$$

Hence, using the representation (132) and making some simple transformations, we obtain

$$\frac{\partial T}{\partial p'^2} (p^2,\ p'^2;\ 0)\Big|_{p'=p^2} = \frac{\partial T}{\partial p'^2} (p'^2,\ p^2;\ 0)\Big|_{p'^2=p^2}$$

$$= (p^2 - m^2_{\pi}) \int da^2\, \frac{\rho\, (a^2)}{(p^2 - a^2 + \mathrm{i}\varepsilon)^3} \{ (a^2 - m^2_{\pi})\, [(1 - l_{\varphi})\, p^2 - (3 + l_{\varphi})\, a^2] - (1 + l_{\varphi})\, (p^2 - a^2)^2 \}. \qquad (138)$$

The right-hand equality in (138) follows from the property of crossing symmetry. In particular, as $p^2 \to m^2_{\pi}$ and $p^2 \to 0$,

$$\frac{\partial T}{\partial p^2} (m^2_{\pi},\ p^2;\ 0)\Big|_{p^2=m^2_{\pi}} = \frac{\partial T}{\partial p^2} (p^2,\ m^2_{\pi};\ 0)\Big|_{p^2=m^2_{\pi}} = -(l_{\varphi} + 1); \qquad (139)$$

$$\frac{\partial T}{\partial p^2} (0,\ p^2;\ 0)\Big|_{p^2=0} = \frac{\partial T}{\partial p^2} (p^2,\ 0;\ 0)\Big|_{p^2=0}$$

$$= -(l_{\varphi} + 1) - (2l_{\varphi} + 4)\, m^2_{\pi} \int da^2\, \frac{\sigma\, (a^2)}{a^2}$$

$$+ (3 + l_{\varphi})\, m^4_{\pi} \int da^2\, \frac{\sigma\, (a^2)}{a^4}. \qquad (140)$$

We now estimate the mean square radius of the pion, which is determined by the derivative of the mass form factor at $t = 0$:

$$\overline{r^2_0}/6 = \theta'\, (m^2_{\pi},\ m^2_{\pi},\ 0)/\theta\, (m^2_{\pi},\ m^2_{\pi};\ 0)$$
$$= T'\, (m^2_{\pi},\ m^2_{\pi};\ 0)/T\, (m^2_{\pi},\ m^2_{\pi};\ 0). \qquad (141)$$

The second equality in (141) follows from (130).

We consider the matrix element

$$\mathrm{i} \int d^4x\, \exp\, (\mathrm{i}px)\, \theta\, (x_0)\, \langle P\, (q)\, |\, [J^A_{\mu 3}\, (x),\ \theta_{0\nu}\, (0)]\, |\, P\, (q')\rangle,$$

where $P(q)$ is a proton with momentum q, and $J^A_{\mu 3}$ is the third isotropic component of the axial current. By the standard method using the reduction technique and PCAC:

$$\partial^{\mu} J^A_{\mu a} = \frac{m^2_{\pi} f_{\pi}}{\sqrt{2}}\, \varphi_a$$

we arrive at the inequality

$$\frac{\mathrm{i} f_{\pi}}{\sqrt{2}}\, \underset{\text{out}}{\langle P\, (q)\, \pi^0\, (p)}\, |\, \theta_{0\mu}\, (0)\, |\, P\, (q')\rangle$$

$$= -\int d^4x\, \exp\, [\mathrm{i}x\, (q' - q)\, \delta\, (x_0)]\, \langle P\, (q)\, |\, [\theta_{0\mu}\, (x),\ J^A_{03}\, (0)]\, |\, P\, (q')\rangle. \qquad (142)$$

Using the equal-time commutators[27]

$$[\theta_{00}\, (x),\ J^A_{0a}\, (y)]_{x_0=y_0} = -\mathrm{i}\partial^0 J^A_{0a}\, (y)\, \delta\, (\mathbf{x} - \mathbf{y}) - \mathrm{i} J^A_{ka}\, (y)\, \partial^k \delta\, (\mathbf{x} - \mathbf{y});$$
$$[\theta_{0i}\, (x),\ J^A_{0a}\, (y)]_{x_0=y_0} = -\mathrm{i}\partial_i J^A_{0a}\, (y)\, \delta\, (\mathbf{x} - \mathbf{y}) - \frac{1}{3}\, \mathrm{i} l_{J_0} J^A_{0a}\, (y)\, \partial_i \delta\, (\mathbf{x} - \mathbf{y}) \qquad (143)$$

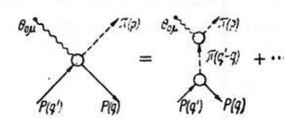
and ignoring the second-order Schwinger terms, we find that

$$\left.\begin{aligned} \frac{\mathrm{i} f_{\pi}}{\sqrt{2}}\, \underset{\substack{\text{out}\\ p\to\infty}}{\langle P\, (q)\, \pi^0\, (p)}\, |\, \theta_{00}\, (0)\, |\, P\, (q')\rangle &= \mathrm{i}\, \langle P\, (q)\, |\, \partial^{\mu} J^A_{\mu 3}\, (0)\, |\, P\, (q')\rangle \\ &= \frac{\mathrm{i} m^2_{\pi}}{\sqrt{2}}\, f_{\pi}\, \langle P\, (q)\, |\, \varphi_{\pi^0}\, (0)\, |\, P\, (q')\rangle; \\ \frac{\mathrm{i} f_{\pi}}{\sqrt{2}}\, \underset{\substack{\text{out}\\ p\to\infty}}{\langle P\, (q)\, \pi^0\, (p)}\, |\, \theta_{0i}\, (0)\, |\, P\, (q')\rangle & \\ &= \frac{1}{3}\, (l_{J_0} + 3)\, (q' - q)\, \langle P(q)\, |\, J^A_{03}\, (0)\, |\, P\, (q')\rangle. \end{aligned}\right\} \qquad (144)$$

Assuming that the dimension l_{J_0} of the current takes a value close to the canonical value, i.e., $l_{J_0} \approx -3$, from (144) we obtain

$$\underset{\substack{\text{out}\\ p\to\infty}}{\langle P\, (q)\, \pi^0\, (p)}\, |\, \theta_{0\mu}\, (0)\, |\, P\, (q')\rangle = g_{0\mu}\, \frac{m^2_{\pi}}{-t + m^2_{\pi}}\, \langle P\, (q)\, |\, \eta_{\phi}\, (0)\, |\, P\, (q')\rangle, \qquad (145)$$

where η_{ϕ} is the source of the π^0-meson field: $(\Box m^2_{\pi})\, \phi = \eta_{\phi}$. We now separate the total term with respect to t in the matrix element $\langle P(q)\, \pi^0(p)\, |\, \theta_{0\mu}(0)\, |\, P(q')\rangle$. In diagrammatic language, this means



It is readily seen that

$$\underset{\substack{\text{out}\\ p\to\infty}}{\langle P\, (q)\, \pi^0\, (p)}\, |\, \theta_{0\mu}\, (0)\, |\, P\, (q')\rangle$$

$$= \frac{1}{-t + m^2_{\pi}}\, \langle P\, (q)\, |\, \eta_{\phi}\, (0)\, |\, P\, (q')\rangle\, \langle \pi^0\, (p)\, |\, \theta_{0\mu}\, (0)\, |\, \pi^0\, (q' - q)\rangle + \ldots, \qquad (146)$$

where the ellipsis stands for terms that do not contain poles as $t = m^2_{\pi}$. It follows from (145) and (146) that

$$\langle \pi^0\, (p)\, |\, \theta_{0\mu}\, (0)\, |\, \pi^0\, (q' - q)\rangle|_{(q' - q)^2 = m^2_{\pi}} = g_{0\mu} m^2_{\pi}. \qquad (147)$$

Equation (147) can be used to calculate $\theta(0, m^2_{\pi}; m^2_{\pi})$. Indeed, the matrix element $\langle \pi^0(p)\, |\, \theta_{\mu\nu}(0)\, |\, \pi(p')\rangle$ has the form

$$\langle \pi^0\, (p)\, |\, \theta_{\mu\nu}\, (0)\, |\, \pi^0\, (p')\rangle = c_1 g_{\mu\nu}$$
$$+ c_2 P_{\mu} P_{\nu} + c_3 k_{\mu} k_{\nu} + c_4\, (P_{\mu} k_{\nu} + k_{\mu} P_{\nu}), \qquad (148)$$

where $P \equiv (p' + p)/2$, $k \equiv (p' - p)$, c_i are form factors that depend on the variables p^2 and p'^2, $t \equiv k^2$, and $c_i = c_i(p^2, p'^2; t)$.

Comparing (148) and (147), we obtain

$$\left.\begin{aligned} c_1\, (0,\ m^2_{\pi};\ m^2_{\pi}) &= m^2_{\pi}; \\ c_2\, (0;\ m^2_{\pi};\ m^2_{\pi})/4 + c_3\, (0,\ m^2_{\pi};\ m^2_{\pi}) + c_4\, (0,\ m^2_{\pi};\ m^2_{\pi}) &= 0. \end{aligned}\right\} \qquad (149)$$

It follows from (148) and (149) that

$$\theta\, (0,\ m^2_{\pi};\ m^2_{\pi}) = 4m^2_{\pi}. \qquad (150)$$

This relation together with (129) gives

$$T\, (0,\ m^2_{\pi};\ m^2_{\pi}) = (4 + l_{\phi})\, m^2_{\pi}. \qquad (151)$$

Now, using the expansion formula for $T(p^2, p'^2; t)$ at the point $p^2_0 = p'^2_0 = m^2_{\pi}$, $t_0 = 0$ for $p^2 = 0$, $p'^2 = m^2_{\pi}$, $t = m^2_{\pi}$:

$$T\, (0,\ m^2_{\pi};\ m^2_{\pi}) = T\, (m^2_{\pi},\ m^2_{\varphi};\ 0)$$

$$- m^2_{\pi}\, \frac{\partial T\, (p^2,\ m^2_{\pi};\ 0)}{\partial p^2}\Big|_{p^2=m^2_{\pi}} + m^2_{\pi}\, \frac{\partial T\, (m^2_{\pi},\ m^2_{\pi};\ t)}{\partial t}\Big|_{t=0} + O\, (m^4_{\pi}),$$

and using (134), (139), and (151), we obtain

$$\frac{\partial \theta\, (m^2_{\pi},\ m^2_{\pi};\ t)}{\partial t}\Big|_{t=0} = \frac{\partial T\, (m^2_{\pi},\ m^2_{\pi};\ t)}{\partial t}\Big|_{t=0} = 1 + O\, (m^2_{\pi}) \qquad (152)$$

and therefore

$$\overline{r^2_0} \approx 3/m^2_{\pi}. \qquad (153)$$

Suppose the axial currents $J^A_{\mu a}(x)$ have definite behavior under the scale transformation, i.e.,

$$[D(x_0),\ J^A_{\mu a}(x)] = -\mathrm{i}\,(l_{J_{\mu a}} - x^\nu\partial_\nu)\,J^A_{\mu a}. \tag{154}$$

From (154), we can readily derive the corresponding transformation law for the charge:

$$\left.\begin{aligned}[D(x_0),\ Q^A_a(x_0)] &= -\mathrm{i}\,(l_{J_{0a}}+3)\,Q^A_a(x_0)\\ &\quad + \mathrm{i}x_0\int dx\,\partial^\mu J^A_{\mu a};\\ Q^A_a(x_0) &\equiv \int dx J^A_{0a}(x).\end{aligned}\right\} \tag{155}$$

Differentiating both sides of Eq. (155) with respect to the time, we obtain

$$\begin{aligned}&[\dot D(0),\ Q^A_a(0)] + [D(0),\ \dot Q^A_a(0)]\\ &= -\mathrm{i}\,(l_{J_{0a}}+3)\,\dot Q^A_a(0) + \mathrm{i}\int dx\,\partial^\mu J^A_{\mu a}(0,\mathbf{x}).\end{aligned} \tag{156}$$

Substituting here the expression

$$\dot D(x_0) = -\int dx\,\theta^\mu_\mu(x);\quad \dot Q^A_a(x_0) = \int dx\,\partial^\mu J^A_{\mu a}(x),$$

and using the PCAC hypothesis, we rewrite this in the form

$$\begin{aligned}&-\int\limits_{x_0=0} dx\,[\theta^\mu_\mu(x),\ Q^A_a(0)]\\ &= \frac{\mathrm{i}m_\pi f_\pi}{\sqrt{2}}\int\limits_{x_0=0} dx\,[l_\varphi - l_{J_{0a}} - 2 - x^k\partial_k]\,\varphi_a(x)\ .\end{aligned} \tag{157}$$

We take the matrix element of both sides of Eq. (157) between the single-particle states of the π meson and σ meson. Then after some simple transformations on the right-hand side, we obtain

$$\begin{aligned}&-\int dx\,\langle\sigma(q)\,|\,[\theta^\mu_\mu(x),\ Q^A_a(0)]\,|\,\pi_b(p)\rangle\\ &= \frac{\mathrm{i}m^2_\pi f_\pi}{\sqrt{2}}\,(l_\varphi + l_{J_0} + 1)\,(2\pi)^3\,\delta(\mathbf{q}-\mathbf{p})\,\frac{1}{-(q-p)^2+m^2_\pi}\\ &\quad\times\langle\sigma(q)\,|\,\eta_a(0)\,|\,\pi_b(p)\rangle.\end{aligned} \tag{158}$$

It is interesting to note that the rms radius of the pion obtained in this way is appreciably greater than the rms electromagnetic radius: $(\overline{r^2_0})^{1/2} \approx 4\sqrt{\overline{r^2_{em}}}$.

6. THE $\sigma\pi\pi$ VERTEX

From the behavior of the current under a scale transformation we can draw conclusions about the nature of the matrix elements of the corresponding physical processes. We show here how, proceeding from the scale transformation law for the current, we can obtain an expression for the $\sigma\pi\pi$ coupling constant.[31] Expanding the matrix element on the left-hand side of Eq. (158) with respect to a complete set of intermediate states, we obtain

$$\begin{aligned}&\int dx\,\langle\sigma(q)\,|\,[\theta^\mu_\mu(x),\ Q^A_a(0)]\,|\,\pi_b(p)\rangle\\ &= \sum_m (2\pi)^3\,\{\delta(\mathbf{q}-\mathbf{p}_m)\,\langle\sigma(q)\,|\,\theta^\mu_\mu(0)\,|\,m\rangle\\ &\quad\times\langle m\,|\,Q^A_a(0)\,|\,\pi_b(p)\rangle - \delta(\mathbf{p}-\mathbf{p}_m)\\ &\quad\times\langle\sigma(q)\,|\,Q^A_a(0)\,|\,m\rangle\,\langle m\,|\,\theta^\mu_\mu(0)\,|\,\pi_b(p)\rangle\}.\end{aligned} \tag{159}$$

For the further calculation of this expression, we shall assume that (σ, π) transforms in accordance with the representation $(\frac{1}{2}, \frac{1}{2})$ of the chiral $\overline{SU}(2)\times SU(2)$ group:

$$[Q^A_a,\ \pi_b] = -\mathrm{i}\sigma\delta_{ab};\quad [Q^A_a,\ \sigma] = \mathrm{i}\pi_a \tag{160}$$

and that in the sum on the right-hand side of Eq. (159) the contribution from the lowest possible single-particle state is dominant, namely, that only the state $|m\rangle = |\sigma\rangle$ makes a significant contribution to the first term, and only the state $|m\rangle = |\pi\rangle$ to the second term. Then

$$\begin{aligned}&-\int dx\,\langle\sigma(q)\,|\,[\theta^\mu_\mu(x),\ Q^A_a(0)]\,|\,\pi_b(p)\rangle\\ &= 2\mathrm{i}\,(2\pi)^3\delta(\mathbf{p}-\mathbf{q})\,\delta_{ab}\,(m^2_\sigma - m^2_\pi).\end{aligned} \tag{161}$$

Here, we have used Eq. (97). From (161) and (158), we find

$$\begin{aligned}&\frac{m^2_\pi f_\pi}{\sqrt{2}}\,(l_\varphi - l_{J_0} + 1)\,\frac{1}{-(q-p)^2+m^2_\pi}\,\langle\sigma(q)\,|\,\eta_a(0)\,|\,\pi_b(p)\rangle\\ &= 2\delta_{ab}\,(m^2_\sigma - m^2_\pi).\end{aligned} \tag{162}$$

We determine the coupling constant $G_{\sigma\pi\pi}$ by means of the equation

$$\langle\sigma(q)\,|\,\eta_a(0)\,|\,\pi_b(p)\rangle = \delta_{ab}G_{\sigma\pi\pi}K((q-p)^2), \tag{163}$$

where K is some form factor with the normalization

$$K(m^2_\pi) = 1. \tag{164}$$

Note that the coupling constant $G_{\sigma\pi\pi}$ determined by (163) is equal to the coupling constant G determined by means of the effective $\sigma\pi\pi$ interaction Lagrangian:

$$\mathcal{L}_{\sigma\pi\pi} = G\sigma\pi\pi/2. \tag{165}$$

Using (163), we rewrite Eq. (162) in the form

$$\frac{m^2_\pi f_\pi}{\sqrt{2}}\,(l_\varphi - l_{J_0} + 1)\,G_{\sigma\pi\pi}\,\frac{K((q-p)^2)}{-(q-p)^2 - m^2_\pi}\bigg|_{\mathbf{q}=\mathbf{p}} = 2\,(m^2_\sigma - m^2_\pi). \tag{166}$$

We now go to the limit $|\mathbf{p}| \to \infty$. Then from (166), we obtain

$$G_{\sigma\pi\pi} = \frac{2\sqrt{2}\,(m^2_\sigma - m^2_\pi)}{(l_\varphi - l_{J_0} + 1)\,f_\pi K(0)}. \tag{167}$$

It is natural to assume that the form factor $K(t)$ changes little as t varies. We can then set

$$K(0) \approx K(m^2_\pi) = 1 \tag{168}$$

and obtain the final expression

$$G_{\sigma\pi\pi} = \frac{2\sqrt{2}\,(m^2_\sigma - m^2_\pi)}{(l_\varphi - l_{J_0} + 1)\,f_\pi}. \tag{169}$$

If the dimensions l_{J_0} and l_ϕ have values near the canonical values, i.e., $l_{J_0} \approx -3$ and $l_\phi \approx -1$, then for $m_\sigma \approx 700$ MeV and $f_\pi \approx 0.96m_\pi$ Eq. (169) gives

$$g^2_{\sigma\pi\pi}/4\pi \approx 11, \tag{170}$$

where

$$g_{\sigma\pi\pi} \equiv G_{\sigma\pi\pi}/2m_\pi.$$

It is readily shown that the σ-meson decay width can be expressed in terms of the coupling constant as follows:

$$\begin{aligned}\Gamma_\sigma &\equiv \Gamma_{\sigma\to\pi^+\pi^-} + \Gamma_{\sigma\to\pi^0\pi^0}\\ &= \frac{3}{32\pi}\,\frac{1}{m_\sigma}\,\sqrt{1 - \frac{4m^2_\pi}{m^2_\sigma}}\,G^2_{\sigma\pi\pi},\end{aligned} \tag{171}$$

Substituting here the experimental value $\Gamma_\sigma \approx 400$ MeV, we obtain

$$g^2_{\sigma\pi\pi}/4\pi \approx 10.4. \tag{172}$$

Thus, (170) and (172) agree well with each other within the experimental accuracy.

Equation (153) together with the value of $G_{\sigma\pi\pi}$ found above can be used to estimate the gravitational coupling constant of the σ meson, which is determined by the equation

$$\langle 0\,|\,\theta_{\mu\nu}(0)\,|\,\sigma(k)\rangle = \frac{1}{3}F_\sigma m^2_\sigma\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \tag{173}$$

For this, we assume that

$$\theta(t) \equiv \langle \pi(p) | \theta^{\mu}_{\mu}(0) | \pi(p') \rangle |_{p^2=p'^2=m_\pi^2} \equiv \theta(m_\pi^2, m_\pi^2; t)$$

satisfies the once-subtracted dispersion relation

$$\theta(t) = 2m_\pi^2 + \frac{t}{\pi} \int_{4m_\pi^2}^{\infty} dt' \frac{\mathrm{Im}\,\theta(t')}{t'(t'-t-i\varepsilon)}. \quad (174)$$

Further, we assume that in the expression

$$\mathrm{Im}\,\theta(t') = \frac{1}{2}(2\pi)^4 \sum_n \delta^{(4)}(p_n - p - p')$$
$$\times \langle 0 | \theta^{\mu}_{\mu}(0) | n \rangle \langle n | \eta_\pi(0) | \pi(p) \rangle; \quad t' \equiv (p'+p)^2 \quad (175)$$

we have σ dominance at small t'. Then

$$\mathrm{Im}\,\theta(t') \approx \pi\delta(t' - m_\sigma^2)\, m_\sigma^2 F_\sigma G_{\sigma\pi\pi}, \quad (176)$$

and from (6.21) we obtain

$$\theta(t) \approx 2m_\pi^2 + F_\sigma G_{\sigma\pi\pi} t / (m_\sigma^2 - t). \quad (177)$$

It follows that

$$F_\sigma \approx \frac{m_\sigma^2}{G_{\sigma\pi\pi}} \frac{d\theta(0)}{dt} \approx \frac{m_\sigma^2}{G_{\sigma\pi\pi}} \approx 1.1 f_\pi \approx m_\pi. \quad (178)$$

7. CONNECTION BETWEEN SCALE INVARIANCE AND CHIRAL INVARIANCE

Making a definite assumption about the behavior of the Hamiltonian density under scale and chiral transformations, we can establish a connection between the consequences of scale invariance and chiral invariance. As examples, we can mention the sum rules for the spectral functions.[32,33]

We consider the following propagators for the divergence of the axial currents and for the trace of the energy-momentum tensor:

$$\Delta_a(p^2) \equiv \int d^4x \exp(ipx) \langle 0 | T\{\partial^{\mu} J^A_{\mu a}(x)\, \partial^{\nu} J^A_{\nu a}(0)\} | 0 \rangle; \quad (179)$$

$$\Delta_\theta(p^2) \equiv \int d^4x \exp(ipx) \langle 0 | T\{\theta^{\mu}_{\mu}(x)\, \theta^{\nu}_{\nu}(0)\} | 0 \rangle. \quad (180)$$

The corresponding spectral functions ρ are expressed in terms of them by means of the representation

$$\Delta(p^2) = i \int da^2 \rho(a^2) / (p^2 - a^2 + i\varepsilon). \quad (181)$$

For $p^2 = 0$,

$$\Delta_a(0) = -\langle 0 | [Q^A_a(0), \partial^{\nu} J^A_{\nu a}(0)] | 0 \rangle; \quad (182)$$

$$\Delta_\theta(0) = \langle 0 | [D(0), \theta^{\nu}_{\nu}(0)] | 0 \rangle. \quad (183)$$

The right-hand side of (182) is completely determined by the behavior of the Hamiltonian under chiral transformations, since[35]

$$\partial^{\nu} J^A_{\nu a} = i[\theta_{00}, Q^A_a], \quad (184)$$

and the right-hand side of (183) is determined by the dimensions l_n that occur in the decomposition

$$\theta_{00} = \sum_n \theta_{00}^{(n)}; \quad \theta^{\mu}_{\mu} = \sum_n (l_n + 4)\, \theta_{00}^{(n)}, \quad (185)$$

where $\theta_{00}^{(n)}$ is the component with dimension l_n.

The original variant[35] of the Gell-Mann–Oakes–Renner chiral Hamiltonian has been investigated on many occasions. It has the form

$$\theta_{00} = \tilde{\theta}_{00} + \theta'_{00}; \quad \theta'_{00} = -(u_0 + cu_8), \quad (186)$$

where θ_{00} is invariant under chiral transformations, and u_0 and u_8 are the scalar components of the representation $(3, \bar{3}) + (\bar{3}, 3)$ of the chiral group, i.e.,

$$\left.\begin{aligned}
&[Q_a, u_b] = if_{abc}u_c; \quad [Q_a, u_0] = 0;\\
&[Q^A_a, u_b] = -id_{abc}v_c - i\sqrt{\tfrac{2}{3}}\,\delta_{ab}v_0;\\
&[Q^A_a, u_0] = -i\sqrt{\tfrac{2}{3}}\,v_a;\\
&[Q_a, v_b] = if_{abc}v_c; \quad [Q_a, v_0] = 0;\\
&[Q^A_a, v_b] = id_{abc}u_c + i\sqrt{\tfrac{2}{3}}\,\delta_{ab}u_0;\\
&[Q^A_a, v_0] = i\sqrt{\tfrac{2}{3}}\,u_a,
\end{aligned}\right\}$$

where f_{abc} and d_{abc} are constants which occur in the commutators and anticommutators of the Gell-Mann matrices[34]:

$$[\lambda_a/2, \lambda_b/2] = if_{abc}\lambda_c/2;$$
$$\{\lambda_a/2, \lambda_b/2\} = d_{abc}\lambda_c/2 + \delta_{ab}/3.$$

Recently, however, some difficulties in the model (186) have been noted, these being due, in particular, to the absence of a component with isospin 2 in the so-called σ term. In this connection, models are now considered in which terms from other representations are added to θ'_{00}. Among these, the most appropriate model seems to be the model[36,37] with

$$\theta'_{00} = -(u_0 + cu_8 + g_8), \quad (188)$$

where g_8 is the scalar compoent of the representation $(1, 8) + (8, 1)$, i.e.,

$$\left.\begin{aligned}
&[Q_a, g_b] = if_{abc}g_c; \quad [Q_a, h_b] = if_{abc}h_c;\\
&[Q^A_a, g_b] = if_{abc}h_c; \quad [Q^A_a, h_b] = if_{abc}g_c,
\end{aligned}\right\} \quad (189)$$

and also the model[38,39] with

$$\theta'_{00} = -(u_0 + cu_8 + \varphi), \quad (190)$$

where

$$\varphi = \sum_{a=1}^{8} Z_{aa} - \frac{4}{3}\sum_{i=1}^{3} Z_{ii} - 4Z_{88} \quad (191)$$

behaves like a term of the 27-plet with $T = Y = 0$ under $SU(3)$ transformations, and Z_{ab} transforms in accordance with the representation (8, 8) of the chiral group, i.e.,

$$\left.\begin{aligned}
&[Q_a, Z_{bc}] = i(f_{abd}Z_{dc} + f_{acd}Z_{bd});\\
&[Q^A_a, Z_{bc}] = i(f_{abd}Z_{dc} - f_{acd}Z_{bd}).
\end{aligned}\right\} \quad (192)$$

We consider the case when the part of the Hamiltonian that breaks the chiral symmetry has the form (188). Calculation in accordance with the expressions (182), (184), and (187) gives

$$\Delta_\pi(0) = -\frac{\sqrt{2}+c}{3}\, i \langle 0 | u_0 + \sqrt{2}\, u_8 | 0 \rangle; \quad (193)$$

$$\Delta_K(0) = \frac{2\sqrt{2}-c}{12}\, i \langle 0 | -2\sqrt{2}\, u_0 + u_8 | 0 \rangle - \frac{3}{4}\, i \langle 0 | g_8 | 0 \rangle. \quad (194)$$

Suppose further that $\tilde{\theta}_{00}$ can be represented as a sum of terms $\tilde{\theta}_{00}^{(n)}$ with dimensions l_n, and that u and g also have definite dimensions l_u and l_g. Then from (183) and (185) we have

$$\Delta_\theta(0) = -i \sum_n l_n (l_n + 4) \langle 0 | \tilde{\theta}_{00}^{(n)} | 0 \rangle + il_u (l_u + 4) \langle 0 | u_0 - cu_8 | 0 \rangle$$
$$+ il_g (l_g + 4) \langle 0 | g_8 | 0 \rangle. \quad (195)$$

We also use the equations

$$\langle 0 | \theta_{00} | 0 \rangle = 0 = \langle 0 | \theta^{\mu}_{\mu} | 0 \rangle,$$

which follow from the requirement that the vacuum be Lorentz invariant.

Note that the $\tilde{\theta}_{00}^{(n)}$ must include at least one term with nonvanishing vacuum expectation value, since, as is readily seen, we should otherwise have $\Delta_\theta(0)=0$. Of interest are the following special cases, when from (193)–(195) it is possible to obtain a relation between $\Delta_\pi(0)$, $\Delta_k(0)$, and $\Delta_\theta(0)$ and, therefore, a sum rule for the spectral functions ρ_π, ρ_k, and ρ_θ:

A. All the l_n in (193) are equal to 4, i.e., $\tilde{\theta}_{00}$ preserves simultaneously the chiral and the scale invariance. Then

$$(c+\sqrt{2})\,\Delta_\theta(0) = \frac{(l_n+4)(l_g+4)(l_g-l_n)}{\sqrt{2}(2\sqrt{2}-c)\,l_g+4(1+2\sqrt{2}c)-3(1-\sqrt{2}c)\,l_n} \times [(2\sqrt{2}-c)(2\sqrt{2}c+1)\Delta_\pi(0)+4(\sqrt{2}+c)(1-\sqrt{2}c)\Delta_K^{(0)}]. \quad (196)$$

B. Among the $\tilde{\theta}_{00}^{(n)}$ there is only one term, which we denote by δ, with nonvanishing vacuum expectation value. Then

$$(c+\sqrt{2})\,\Delta_\theta(0) = \frac{(l_n-l_\delta)(l_g-l_\delta)(l_g-l_n)}{\sqrt{2}(2\sqrt{2}-c)\,l_g-(1+2\sqrt{2}c)\,l_\delta-3(1-\sqrt{2}c)\,l_n} \times [(2\sqrt{2}-c)(2\sqrt{2}c+1)\Delta_\pi(0)+4(\sqrt{2}+c)(1-\sqrt{2}c)\Delta_K(0)]. \quad (197)$$

We now consider the case when the chiral symmetry breaking part of the Hamiltonian has the form (190). Calculation in accordance with Eqs. (182), (184), and (192) gives

$$\Delta_\pi(0) = -\frac{\sqrt{2}+c}{3}\, i\langle 0|\sqrt{2}u_0+u_8|0\rangle + 4i\left\langle 0\left|\tfrac{2}{3}Z_{11}-Z_{44}\right|0\right\rangle; \quad (198)$$

$$\Delta_K(0) = \frac{2\sqrt{2}-c}{12}\, i\langle 0|-2\sqrt{2}u_0+u_8|0\rangle + i\langle 0|-Z_{11}+2Z_{44}+3Z_{88}|0\rangle. \quad (199)$$

We have used here the equations

$$\langle Z_{11}\rangle_0 = \langle Z_{22}\rangle_0 = \langle Z_{33}\rangle_0; \quad (200)$$

$$\langle Z_{44}\rangle_0 = \langle Z_{55}\rangle_0 = \langle Z_{66}\rangle_0 = \langle Z_{77}\rangle_0, \ldots, \quad (201)$$

which follow from the isotropic invariance of the vacuum. Instead of (195), we now have

$$\Delta_\theta(0) = -i\sum_n l_n(l+4)\langle 0|\tilde{\theta}_{00}^{(n)}|0\rangle + il_u(l_u+4)\langle 0|u_0+cu_8|0\rangle + il_\varphi(l_\varphi+4)\langle 0|\varphi|0\rangle. \quad (202)$$

To deduce a relationship between $\Delta_\pi(0)$, $\Delta_K(0)$, and $\Delta_\theta(0)$ from (198), (199), and (202), we make the assumption that in calculating (198), (199), and (202) we can take the vacuum to be invariant under SU(3) transformations. This is equivalent to ignoring in these equations the corrections due to the noninvariance of the vacuum under $SU(3)$ transformations with respect to the other terms which are present. We can then set

$$\langle Z_{11}\rangle_0 \approx \langle Z_{44}\rangle_0 \approx \langle Z_{88}\rangle_0 \equiv Z;$$
$$\langle u_8\rangle_0 \approx 0$$

and rewrite these equations in the form

$$\Delta_\pi(0) = -\frac{\sqrt{2}(\sqrt{2}+c)}{3}\, i\langle u_0\rangle_0 - \frac{4}{3}iZ; \quad (203)$$

$$\Delta_K(0) = -\frac{\sqrt{2}(2\sqrt{2}-c)}{6}\, i\langle u_0\rangle_0 + 4iZ; \quad (204)$$

$$\Delta_\theta(0) = -i\sum_n l_n(l_n+4)\langle\tilde{\theta}_{00}^{(n)}\rangle_0 + il_u(l_u+4)\langle u_0\rangle_0. \quad (205)$$

For the same reason as above, the $\tilde{\theta}_{00}^{(n)}$ must include at least two terms with nonvanishing vacuum expectation value, and also one term with dimension $l\neq -4$ with nonvanishing vacuum expectation value.

A sum rule now exists in the following special cases:

C. The $\tilde{\theta}_{00}^{(n)}$ include only one term, which we denote by δ, with dimension $l_\delta \neq -4$, i.e., it has the form

$$\theta_{00} = \tilde{\tilde{\theta}}_{00} + \delta - (u_{00}+cu_8+\varphi), \quad (206)$$

where $\tilde{\theta}_{00}$ preserves simultaneously the chiral and the scale invariance, and δ preserves only the chiral invariance. Then

$$\Delta_\theta(0) = \frac{6}{16+5\sqrt{2}c}(l_\delta-l_u)(l_u+4)[3\Delta_\pi(0)+\Delta_K(0)]. \quad (207)$$

D. The $\tilde{\theta}_{00}^{(n)}$ include only two terms, which we denote by δ_1 and δ_2, with nonvanishing vacuum expectation value. We have

$$\Delta_\theta(0) = \frac{6}{16+5\sqrt{2}c}(l_{\delta_1}-l_u)(l_u-l_{\delta_2})[3\Delta_\pi(0)+\Delta_K(0)]. \quad (208)$$

Finally, we rewrite the results (196), (197), (207), and (208) in the approximation of complete dominance, i.e., for

$$\left.\begin{aligned} \rho_\theta(a^2) &\approx (F_\sigma m_\sigma^2)^2\,\delta(a^2-m_\sigma^2);\\ \rho_\pi(a^2) &\approx (f_\pi m_\pi^2/\sqrt{2})^2\,\delta(a^2-m_\pi^2);\\ \rho_K(a^2) &\approx (f_K m_K^2/\sqrt{2})^2\,\delta(a^2-m_K^2), \end{aligned}\right\} \quad (209)$$

where F_σ is the gravitational coupling constant of the σ meson, $f_\pi\cos\theta$ and $f_K\sin\theta$ are the decay constants of the π and K meson, and θ is the Cabibbo angle.

We obtain, respectively,

$$(\sqrt{2}+c)F_\sigma^2 m_\sigma^2 = \frac{(l_u+4)(l_g+4)(l_g-l_u)}{\sqrt{2}(2\sqrt{2}-c)\,l_g-(1+2\sqrt{2}c)\,l_\delta-3(1-\sqrt{2}c)\,l_u} \times\left[\tfrac{1}{2}(2\sqrt{2}-c)(2\sqrt{2}c+1)f_\pi^2 m_\pi^2 + 2(\sqrt{2}+c)(1-\sqrt{2}c)f_K^2 m_K^2\right]; \quad (210)$$

$$(\sqrt{2}+c)F_\sigma^2 m_\sigma^2 = \frac{(l_u-l_\delta)(l_g-l_\delta)(l_g-l_u)}{\sqrt{2}(2\sqrt{2}-c)\,l_g-(1+2\sqrt{2}c)\,l_\delta-3(1-\sqrt{2}c)\,l_u} \times\left[\tfrac{1}{2}(2\sqrt{2}-c)(2\sqrt{2}c-1)f_\pi^2 m_\pi^2 + 2(\sqrt{2}+c)(1-\sqrt{2}c)f_K^2 m_K^2\right]; \quad (211)$$

$$(16+5\sqrt{2}c)F_\sigma^2 m_\sigma^2 = 3(l_\delta-l_u)(l_u+4)[3f_\pi^2 m_\pi^2+f_K^2 m_K^2]; \quad (212)$$

$$(16+5\sqrt{2}c)F_\sigma^2 m_\sigma^2 = 3(l_{\delta_1}-l_u)(l_u-l_{\delta_1})[3f_\pi^2 m_\pi^2+f_K^2 m_K^2]. \quad (213)$$

These results can be used to obtain information about the dimensions of the terms in θ_{00} in any particular considered model.[33]

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Translated by Julian B. Barbour